Asymptotics of the Stirling numbers of the second kind revisited: A saddle point approach

Guy Louchard

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Outline

Introduction

Let $\left\{ {n \atop m} \right\}$ be the Stirling number of the second kind. Their generating function is given by

$$\sum_n \frac{m!}{n!} \left\{ {n \atop m} \right\} z^n = f(z)^m,$$

$$f(z) := e^z - 1.$$

In the sequel all asymptotics are meant for $n \to \infty$.
Let us first summarize the related litterature. The asymptotic Gaussian approximation in the central region is proved in Harper [7]. See also Bender [1], Sachkov [13] and Hwang [10].
In the non-central region, most of the previous papers use the solution of

$$\frac{\rho e^{\rho}}{e^{\rho} - 1} = \frac{n}{m}. \tag{1}$$

As shown in the next section, this actually corresponds to a Saddle point.

Let us mention

- Hsu [8]:
  For $t = o(n^{1/2})$

$$\left\{ {n+t \atop n} \right\} = \frac{n^{2t}}{2^t t!} \left[1 + \frac{f_1(t)}{n} + \frac{f_2(t)}{n^2} + \dots \right],$$

$$f_1(t) = \frac{1}{3} t(2t+1).$$

- Moser and Wyman [12]:
  For $t = o(\sqrt{n})$,

$$\left\{ {n \atop n-t} \right\} = \binom{n}{t} q^{-t} \left[1 + \frac{(t)_2}{12} q + \frac{(t)_2}{288} q^2 + \dots \right],$$

$$q = \frac{2}{n-t}.$$

For $n - m \to \infty, n \to \infty$,

$$\left\{ {n \atop m} \right\} = \frac{n!(e^{\rho} - 1)^m}{2\rho^n m!(\pi m \rho H)^{1/2}} \left[1 - \frac{1}{m\rho} \left(\frac{15 C_3^2}{16 \rho^2 H} - \frac{3 C_4}{4 \rho H^2} \right) + \dots \right],$$

$$H = \frac{e^{\rho}(e^{\rho} - 1 - \rho)}{2(e^{\rho} - 1)^2},$$

C_3, C_4 are functions of ρ.

- Good [6]:

$$\left\{ {n+t \atop t} \right\} = \frac{(t+n)!(e^{\rho}) - 1)^t}{t!\rho^{t+n}\left[2\pi t\left(1+\kappa-(1+\kappa)^2 e^{-\rho}\right)\right]^{1/2}} \times$$

$$\times \left[1 + \frac{g_1(\kappa)}{t} + \frac{g_2(\kappa)}{t^2} + \dots\right],$$

$$\kappa := \frac{n}{t},$$

$$g_1(\kappa) = \frac{3\lambda_4 - 5\lambda_3^2}{24},$$

$$\lambda_i = \kappa_i(\rho)/\sigma^i,$$

$$\sigma = \kappa_2(\rho)^{1/2},$$

$$\kappa_1 = \kappa, \kappa_2 = (\kappa_1 + 1)(\rho - \kappa_1).$$

- Bender [1]:

$$\left\{ {n \atop m} \right\} \sim \frac{n! e^{-\alpha m}}{m! \rho^{n-1}(1+e^{\alpha})\sigma\sqrt{2\pi n}},$$
$$\frac{n}{m} = (1+e^{\alpha})\ln(1+e^{-\alpha}),$$
$$\rho = \ln(1+e^{-\alpha}),$$
$$\sigma^2 = \left(\frac{m}{n}\right)^2 \left[1 - e^{\alpha}\ln(1+e^{-\alpha})\right].$$

It is easy to see that ρ here coincides with the solution of (1). Bender's expression is similar to Moser and Wyman' result.

- Bleick and Wang [2]:
  Let ρ_1 be the solution of

$$\frac{\rho_1 e^{\rho_1}}{e^{\rho_1} - 1} = \frac{n+1}{m}.$$

Then

$$\left\{ {n \atop m} \right\} = \frac{n!(e^{\rho_1} - 1)^m}{(2\pi(n+1))^{1/2} m! \rho_1^n (1-G)^{1/2}} \times$$
$$\times \left[1 - \frac{A}{24(n+1)(1-G)^3} + \mathcal{O}(1/n^2) \right],$$
$$A := 2 + 18G - 20G^2(e^{\rho_1} + 1)$$
$$+ 3G^3(e^{2\rho_1} + 4e^{\rho_1} + 1) + 2G^4(e^{2\rho_1} - e^{\rho_1} + 1),$$
$$G = \frac{\rho_1}{e^{\rho_1} - 1}.$$

The series is convergent for for $m = o(n^{2/3})$.

- Temme [15]:

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = e^{A} m^{n-m} \binom{n}{m} \sum_{k=0}^{\infty} (-1)^k f_k(t_0) m^{-k},$$

$$f_0(t_0) = \left(\frac{t_0}{(1+t_0)(\rho - t_0)} \binom{n}{m} \right)^{1/2},$$

$$t_0 = \frac{n}{m} - 1,$$

where A is a function of ρ, n, m.

- Tsylova [16]:
  Let $m = tn + o(n^{2/3})$.

$$\left\{ {n \atop m} \right\} = \frac{(\gamma n)^n}{\sqrt{2\pi\delta n}(\gamma n)^m} \exp\left[-(m-tn)^2/(2\delta n)\right](1+o(1)),$$

$$\gamma(1 - e^{-1/\gamma}) = \gamma,$$

$$\delta = e^{-1/\gamma}(t - e^{-1/\gamma}).$$

After some algebra, this coincides with Moser and Wyman' result.

- Chelluri, Richmond and Temme [3]:
  They prove, with other techniques, that Moser and Wyman expression is valid if $n - m = \Omega(n^{1/3})$ and that Hsu formula is valid for $y - x = o(n^{1/3})$
- Erdos and Szekeres: see Sachkov [13], p.164:
  Let $m < n/\ln n$,

$$\left\{ {n \atop m} \right\} = \frac{m^n}{m!} \exp\left[\left(\frac{n}{m} - m\right) e^{-n/m}\right](1 + o(1)).$$

All these papers simply use ρ as the solution of (1). They don't compute the detailed dependence of ρ on α for our range, neither the precise behaviour of functions of ρ they use. Moreover, most results are related to the case $\alpha < 1/2$.

We will use multiseries expansions: multiseries are in effect power series (in which the powers may be non-integral but must tend to infinity) and the variables are elements of a scale: details can be found in Salvy and Shackell [14]. The scale is a set of variables of increasing order. The series is computed in terms of the variable of maximum order, the coefficients of which are given in terms of the next-to-maximum order, etc. Actually we implicitly used multiseries in our analysis of Stirling numbers of the first kind in [11].

Let us finally mention that Hsu [9] consider some generalized Stirling numbers.

In Sec.2, we revisit the asymptotic expansion in the central region and in Sec.3, we analyse the non-central region $j = n - n^{\alpha}, \quad \alpha > 1/2$. We use Cauchy's integral formula and the saddle point method.

Central region

Consider the random variable J_n, with probability distribution

$$\mathbb{P}[J_n = m] = Z_n(m),$$

$$Z_n(m) := \frac{\left\{ n \atop m \right\}}{B_n},$$

where B_n is the nth Bell number. The mean and variance of J_n are given by

$$M := \mathbb{E}(J_n) = \frac{B_{n+1}}{B_n} - 1,$$

$$\sigma^2 := \mathbb{V}(J_n) = \frac{B_{n+2}}{B_n} - \frac{B_{n+1}}{B_n} - 1.$$

Let ζ be the solution of

$$\zeta e^{\zeta} = n.$$

This immediately leads to

$$\zeta = W(n),$$

where W is the Lambert function (we use the principal branch, which is analytic at 0). We have the well-known asymptotic

$$\zeta = \ln(n) - \ln\ln(n) + \frac{\ln\ln(n)}{\ln(n)} + \mathcal{O}(1/\ln(n)^2). \tag{2}$$

To simplify our expressions in the sequel, let

$$F := e^{\zeta},$$
$$G := e^{\zeta/2}.$$

The multiseries' scale is here $\{\zeta, G\}$.

Our result can be summarized in the following local limit theorem

Theorem 2.1

Let $x = (m - M)/\sigma$. Then

$$Z_n(m) = \frac{\left\{ n \atop m \right\}}{B_n}$$
$$= e^{-x^2/2} \frac{(1+\zeta)^{1/2}}{\sqrt{2\pi} G} \left[1 + \frac{x(-6\zeta + 2x^2\zeta + x^2 - 3)}{6G(1+\zeta)^{3/2}} + \mathcal{O}(1/G^2) \right].$$

Proof. By Salvy and Shackell [14], we have

$$M = F + A_1 + \mathcal{O}(1/F),$$
$$\sigma^2 = \frac{F}{1+\zeta} + A_3 + \mathcal{O}(1/F),$$
$$\frac{B_n}{n!} = \exp(T_1) H_0, \tag{3}$$
$$T_1 = -\ln(\zeta)\zeta F + F - \zeta/2 - \ln(\zeta) - 1 - \ln(2\pi)/2, \tag{4}$$

$$A_1 = -\frac{2 + 3/\zeta + 2/\zeta^2}{2(1+1/\zeta)^2},$$

$$A_3 = -\frac{2 + 8/\zeta + 11/\zeta^2 + 9/\zeta^3 + 2/\zeta^4}{2(1+1/\zeta)^4},$$

$$H_0 = \frac{1}{(1+1/\zeta)^{1/2}}\left[1 + A_5/F + \mathcal{O}(1/F^2)\right],$$

$$A_5 = -\frac{2 + 9/\zeta + 16/\zeta^2 + 6/\zeta^3 + 2/\zeta^4}{24(1+1/\zeta)^3}.$$

This leads to (from now on, we only provide a few terms in our expansions, but of course we use more terms in our computations), using expansions in G,

$$\sigma = \frac{G}{(1+\zeta)^{1/2}} + \frac{A_3(1+\zeta)^{1/2}}{2G} + \mathcal{O}(1/G^3),$$
$$\sigma \sim \frac{G}{\sqrt{\zeta}} \sim \frac{\sqrt{n}}{\ln(n)}.$$

We now use the Saddle point technique (for a good introduction to this method, see Flajolet and Sedgewick [4], ch. *VIII*). Let ρ be the saddle point and Ω the circle $\rho e^{\mathbf{i}\theta}$. By Cauchy's theorem,

$$
\begin{aligned}
Z_n(m) &= \frac{n!}{m!B_n}\frac{1}{2\pi\mathbf{i}}\int_{\Omega}\frac{f(z)^m}{z^{n+1}}dz \\
&= \frac{n!}{m!B_n\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{\mathbf{i}\theta})^m e^{-n\mathbf{i}\theta}d\theta \\
&= \frac{n!}{m!B_n\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{m\ln(f(\rho e^{\mathbf{i}\theta}))-n\mathbf{i}\theta}d\theta \\
&= \frac{n!}{m!B_n\rho^n}\frac{f(\rho)^m}{2\pi}\int_{-\pi}^{\pi}\exp\left[m\left\{-\frac{1}{2}\kappa_2\theta^2-\frac{\mathbf{i}}{6}\kappa_3\theta^3+\dots\right\}\right],
\end{aligned}
\tag{5}
$$

$$
\kappa_i(\rho) = \left(\frac{\partial}{\partial u}\right)^i \ln(f(\rho e^u))\Big|_{u=0}. \tag{6}
$$

See Good [5] for a neat description of this technique.

Let us now turn to the Saddle point computation. ρ is the root (of smallest module) of

$$m\rho f'(\rho) - nf(\rho) = 0, \text{ i.e.}$$
$$\frac{\rho e^{\rho}}{e^{\rho} - 1} = \frac{n}{m},$$

which is, of course identical to (1). After some algebra, this gives

$$\rho = \frac{n}{m} + W\left(-\frac{n}{m}e^{-n/m}\right).$$

In the central region, we choose

$$m = M + \sigma x = F + \frac{x}{(1+\zeta)^{1/2}} G + A_1 + \frac{x A_3 (1+\zeta)^{1/2}}{2G} + \mathcal{O}(1/G^2).$$

This leads to

$$\ln(m) = \zeta + \frac{x}{(1+\zeta)^{1/2} G} + \mathcal{O}(1/G^2),$$

$$\frac{n}{m} = \zeta - \frac{\zeta x}{(1+\zeta)^{1/2} G} + \frac{-A_1 \zeta + \zeta x^2/(1+\zeta)}{G^2} + \mathcal{O}(1/G^3),$$

$$\rho = \zeta - \frac{\zeta x}{(1+\zeta)^{1/2} G} + \frac{\zeta(-A_1 + x^2/(1+\zeta) - 1)}{G^2} + \mathcal{O}(1/G^3),$$

$$\ln(\rho) = \ln(\zeta) - \frac{x}{(1+\zeta)^{1/2} G} + \mathcal{O}(1/G^2).$$

Now we note that

$$
\begin{aligned}
e^{\rho} - 1 &= \rho e^{\rho} \frac{m}{n}, \\
\ln\left(e^{\rho} - 1\right) &= \rho + \ln(\rho) + \ln(m) - \ln(n), \qquad (7) \\
\ln(n) &= \zeta + \ln(\zeta),
\end{aligned}
$$

$$(8)$$

so, by Stirling's formula, with (4), the first part of (5) leads to

$$\frac{n!}{m! B_n \rho^n} f(\rho)^m = \exp\left[T_2\right] H_1 H_2,$$

$$T_2 = m(\rho + \ln(\rho) - \zeta - \ln(\zeta))$$

$$-(-m + \ln(2\pi)/2 + \ln(m)/2) - \zeta F \ln(\rho) - T_1,$$

$$H_1 = 1/H_0 = (1 + 1/\zeta)^{1/2} - \frac{A_5(1+1/\zeta)^{1/2}}{G^2} + \mathcal{O}(1/G^4),$$

$$H_2 = 1 \bigg/ \left[1 + \frac{1}{12m} + \frac{1}{288m^2} + \mathcal{O}(1/m^3)\right]$$

$$= 1 - \frac{1}{12G^2} + \frac{x}{12G^3(1+\zeta)^{1/2}} + \mathcal{O}(1/G^4).$$

Note carefully that there is a cancellation of the term $m\ln(m)$ in T_2. Using all previous expansions, we obtain

$$\exp(T_2) = e^{-x^2/2+\ln(\zeta)})H_3, \tag{9}$$

$$H_3 = 1 + \frac{x(-15\zeta - 6\zeta^2 - 6A_1 + x^2 - 12A_1\zeta - 6A_1\zeta^2 + 2x^2\zeta - 9}{6(1+\zeta)^{3/2}G}$$

$$+\mathcal{O}(1/G^2).$$

We now turn to the integral in (5). We compute

$$\kappa_2 = -\frac{\rho e^{\rho}(-e^{\rho} + 1 + \rho)}{(e^{\rho} - 1)^2} = \zeta - \frac{\zeta x}{(1+\zeta)^{1/2}G} + \mathcal{O}(1/G^2),$$

and similar expressions for the next κ_i that we don't detail here. Note that $\kappa_3, \kappa_5, \ldots$ are useless for the precision we attain here.

Now we use the classical trick of setting

$$m\left[-\kappa_2\theta^2/2! + \sum_{l=3}^{\infty} \kappa_\ell (\mathbf{i}\theta)^\ell/\ell!\right] = -u^2/2.$$

Computing θ as a series in u, this gives, by inversion,

$$\theta = \frac{1}{G}\sum_{1}^{\infty} a_i u^i,$$

with, for instance

$$a_1 = \frac{1}{\zeta^{1/2}} + \frac{\zeta^{1/2}}{2G^2} + \mathcal{O}(1/G^3).$$

Setting $d\theta = \frac{d\theta}{du} du$, we integrate on $[u = -\infty..\infty]$: this extension of the range can be justified as in Flajolet and Segewick [4], Ch. *VIII*. Now, inserting the term ζ coming in (9) as $e^{\ln(\zeta)}$, this gives

$$H_4 = \frac{\zeta^{1/2}}{\sqrt{2\pi} G}\left(1 + \frac{\zeta}{2G^2} + \mathcal{O}(1/G^3)\right).$$

Finally, combining all expansions,

$$Z_n(m) = \frac{\left\{ n \atop m \right\}}{B_n} = e^{-x^2/2} H_1 H_2 H_3 H_4 = R_1, \tag{10}$$

$$R_1 = e^{-x^2/2} \frac{(1+\zeta)^{1/2}}{\sqrt{2\pi} G}\left[1 + \frac{x(-6\zeta + 2x^2\zeta + x^2 - 3)}{6G(1+\zeta)^{3/2}} + \mathcal{O}(1/G^2)\right].$$

Note that the dominant term is equivalent to the dominant term of $\frac{1}{\sqrt{2\pi}\sigma}$, as expected. More terms in this expression can be obtained if we compute $M, \sigma^2, B_n/n!$ with more precision. Also, using (2), our result can be put into expansions depending on $n, \ln n, \ldots$. ■

To check the quality of our asymptotic, we have chosen $n = 3000$. This leads to

$$\begin{aligned}
\zeta &= 6.184346264\ldots, \\
G &= 22.02488900\ldots, \\
M &= 484.1556441\ldots, \\
\sigma &= 8.156422315\ldots, \\
B_n &= 0.2574879583\ldots 10^{6965}, \\
B_n as &= 0.2574880457\ldots 10^{6965},
\end{aligned}$$

where $B_n as$ is given by (3). Figure 1 shows $Z_n(m)$ and $\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\left(\frac{m-M}{\sigma}\right)^2 \Big/ 2\right]$.

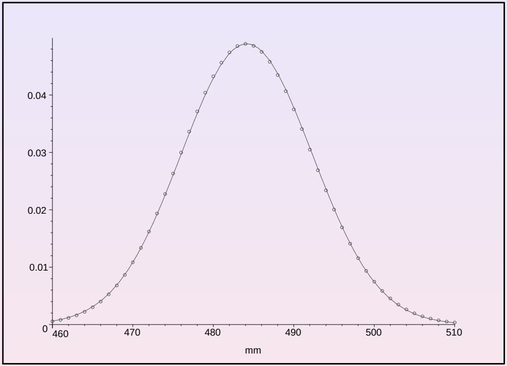

Figure 1: $Z_n(m)$ and $\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{m-M}{\sigma}\right)^2 \Big/ 2\right]$

The fit seems quite good, but to have more precise information, we show in Figure 2 the quotient $Z_n(m) \Big/ \frac{1}{\sqrt{2\pi}\sigma} \exp\left[- \left(\frac{m-M}{\sigma}\right)^2 \Big/ 2\right]$. The precision is between 0.05 and 0.10.

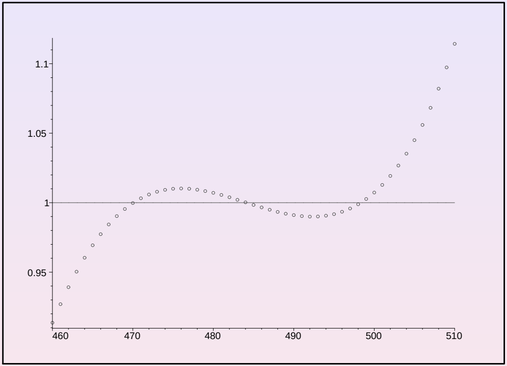

Figure 2: $Z_n(m) \Big/ \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{m-M}{\sigma}\right)^2 \Big/ 2\right]$

Figure 3 shows the quotient $Z_n(m)\,/R_1$. The precision is now between 0.004 and 0.01.

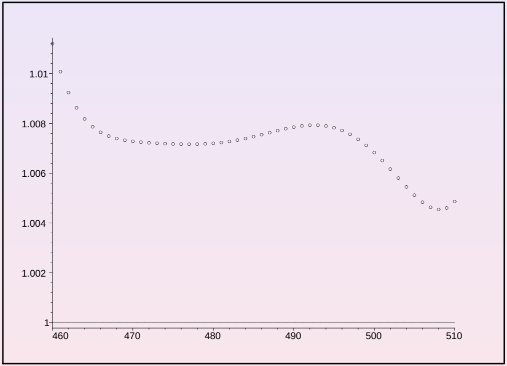

Figure 3: $Z_n(m)\,/R_1$

Large deviation, $m = n - n^\alpha, \quad \alpha > 1/2$

We set

$$\begin{aligned} \varepsilon &:= n^{\alpha-1}, \\ \frac{1}{\varepsilon} &= n^{1-\alpha} \ll n^\alpha \ll n, \\ L &:= \ln(n). \end{aligned}$$

The multiseries' scale is here $\{n^{1-\alpha}, n^\alpha, n\}$.

Our result can be summarized in the following local limit theorem

Theorem 3.1

$$\left\{ {n \atop m} \right\} = e^{T_1} R,$$

$$T_1 = n^{\alpha}(T_{11}L + T_{10}),$$

$$R = \frac{1}{\sqrt{2\pi}\, n^{\alpha/2}} \left[R_0 + \frac{R_1}{n} + \frac{R_2}{n^2} + \mathcal{O}(1/n^3) \right],$$

$$R_0 = R_{00} + \frac{R_{01}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$

$$R_1 = R_{10} + \frac{R_{11}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$

$$R_2 = R_{20} + \frac{R_{21}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$

where $T_{i,j}, R_{i,j}$ are power series in ε.

Proof. Using again the Lambert function, we derive successively (again we only provide a few terms here, we use a dozen of terms in our expansions)

$$
\begin{aligned}
m &= n(1-\varepsilon),\\
\frac{n}{m} &= \frac{1}{1-\varepsilon},\\
\rho &= 2\varepsilon + \frac{4}{3}\varepsilon^2 + \frac{10}{9}\varepsilon^3 + \mathcal{O}(\varepsilon^4),\\
\ln(m) &= L - \varepsilon - \frac{1}{2}\varepsilon^2 + \mathcal{O}(\varepsilon^3),\\
\ln(\rho) &= -L(1-\alpha) + \ln(2) + \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon^2 + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

For the first part of Cauchy's integral, we have, noting that $n\varepsilon = n^{\alpha}$, and using (7),

$$\frac{n!}{m!\rho^n} f(\rho)^m = \exp(T) H_2,$$
$$T = m(\rho + \ln(\rho) - L) - (-m + \ln(m)/2) + (-n + nL + L/2) - n\ln(\rho)$$
$$= T_1 + T_0,$$
$$T_1 = n^{\alpha}(T_{11}L + T_{10}),$$
$$T_{11} = 2 - \alpha,$$
$$T_{10} = 1 - \ln(2) - \frac{4}{3}\varepsilon - \frac{5}{9}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$
$$T_0 = \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$

$$H_1 = \exp(T_0) = 1 + \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$

$$H_2 = \left[1 + \frac{1}{12n} + \frac{1}{288n^2} + \mathcal{O}(1/n^3)\right] \Big/ \left[1 + \frac{1}{12m} + \frac{1}{288m^2} + \mathcal{O}(1/m^3)\right]$$

$$= 1 + \frac{\varepsilon}{12(\varepsilon - 1)n} + \frac{\varepsilon^2}{288(\varepsilon - 1)^2 n^2} + \mathcal{O}(\varepsilon^3/n^3).$$

Note again that there are cancellations, in T_1 of the terms $m \ln(m)$ and $\ln(2\pi)/2$.
Now we turn to the integral part. We obtain, for instance, using (6),

$$\kappa_2 = \varepsilon + \frac{4}{3}\varepsilon^2 + \frac{13}{9}\varepsilon^3 + \mathcal{O}(\varepsilon^4),$$

$$\theta = \frac{1}{\sqrt{n}} \sum_{1}^{\infty} a_i u^i,$$

$$a_1 = \frac{1}{\sqrt{\varepsilon}} \left[1 - \frac{1}{6}\varepsilon^2 - \frac{1}{72}\varepsilon^4 + \mathcal{O}(\varepsilon^6)\right].$$

Integrating, this gives

$$
\begin{aligned}
H_3 &= \frac{1}{\sqrt{2\pi}\, n^{\alpha/2}} \left[H_{31} + \frac{H_{32}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}) \right], \\
H_{31} &= 1 - \frac{1}{6}\varepsilon - \frac{1}{72}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
H_{32} &= -\frac{1}{12} + \frac{1}{72}\varepsilon - \frac{71}{864}\varepsilon^2 + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Now we compute

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = e^{T1} H_1 H_2 H_3 = e^{T_1} R, \tag{11}$$

with

$$R = \frac{1}{\sqrt{2\pi} n^{\alpha/2}} \left[R_0 + \frac{R_1}{n} + \frac{R_2}{n^2} + \mathcal{O}(1/n^3) \right],$$

$$R_0 = R_{00} + \frac{R_{01}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$

$$R_1 = R_{10} + \frac{R_{11}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$

$$R_2 = R_{20} + \frac{R_{21}}{n^{\alpha}} + \mathcal{O}(1/n^{2\alpha}),$$

$$R_{00} = 1 + \frac{1}{3}\varepsilon + \mathcal{O}(\varepsilon^2),$$
$$R_{01} = -\frac{1}{12} - \frac{1}{36}\varepsilon + \mathcal{O}(\varepsilon^2),$$
$$R_{10} = -\frac{1}{12}\varepsilon - \frac{1}{9}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$
$$R_{11} = \frac{1}{144}\varepsilon + \frac{1}{108}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$
$$R_{20} = \frac{1}{288}\varepsilon + \frac{7}{864}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$
$$R_{21} = -\frac{1}{3456}\varepsilon - \frac{7}{10368}\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

Given some desired precision, how many terms must we use in our expansions? It depends on α. For instance, in T_1, $n^{\alpha}\varepsilon^k \gg 1$ if $k < \alpha/(1-\alpha)$. Also ε^k in R_{00} is less than ε^{ℓ}/n in R_{10}/n if $k - \ell > 1/(1-\alpha)$. Any number of terms can be computed by almost automatic computer algebra. We use Maple in this paper.

To check the quality of our asymptotic, we have chosen $n = 100$ and a range $\alpha \in [1/2, 9/10]$, i.e. a range $m \in [37, 90]$. We use 5 or 6 terms in our final expansions. Figure 4 shows the quotient $\left\{ {n \atop m} \right\} / (e^{T_1} R)$. The precision is at least 0.0066. Note that the range $[M - 3\sigma, M + 3\sigma]$, where the Gaussian approximation is useful, is here $m \in [21, 36]$.

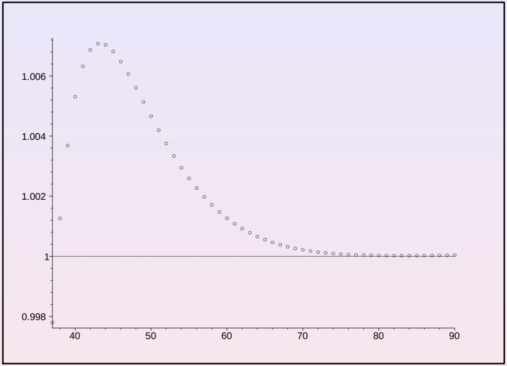

Figure 4: $\left\{ {n \atop m} \right\} / (e^{T_1} R)$

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